AN INFINITE FAMILY OF CUBICS WITH EMERGENT REDUCIBILITY AT DEPTH 1

JASON I. PRESZLER

Abstract. A polynomial $f(x)$ has emergent reducibility at depth n if $f^{\circ k}(x)$ is irreducible for $0 \le k \le n-1$ but $f^{\circ n}(x)$ is reducible. In this paper we prove that there are infinitely many irreducible cubics $f \in \mathbb{Z}[x]$ with $f \circ f$ reducible by exhibiting a one parameter family with this property.

1. Introduction

Given a polynomial $f(x) \in \mathbb{Q}[x]$, one can construct the sequence of iterates $f \circ f(x), f \circ f \circ f(x), \ldots, f^{\circ n}(x)$. Such sequences form dynamical systems and have become the focus of considerable scrutiny in recent years. In the 1980's, Odoni proved fundamental facts about the behavior of the discriminant and resultant [5], proved instances where entire sequences consisted of irreducible polynomials [6], gave examples of sequences with irreducible initial terms but reducible terms after a certain point [6], and showed that the Galois groups of $f^{\circ n}(x)$ embed into the n-fold iterated wreath product of $\mathrm{Gal}(f)$, the Galois group of f [5]. More recently, [1] showed that there are finitely many irreducible quadratic polynomials where $f^{\circ n}$ is reducible if $n \ge 2$. In [2] it was already shown that there are infinitely many irreducible quadratics f with $f \circ f$ reducible. It should be noted that once a term in the sequence is reducible all subsequent terms will be reducible. Additionally, Hindes [4] has shown that the Galois group of $f^{\circ n}$ can fail to be the full n-fold iterated wreath product even when $f^{\circ n}$ is always irreducible.

The phenomena that we focus on in this paper will be called emergent reducibility, which should be thought of as a property of the base polynomial in the dynamical system, as opposed to the term "newly reducible" in [1] which refers to a specific iterate.

Definition 1.1 (Emergent Reducibility at Depth n)**.** *We say a polynomial $f(x) \in K[x]$ has* ***emergent reducibility at depth*** *n if and only if $f^{\circ i}(x)$ is irreducible over $K[x]$ for $0 \le i \le n-1$ and $f^{\circ n}(x)$ is reducible over $K[x]$.*

Note that n counts the number of compositions so $f^{\circ 0}(x) = f(x)$. This notation is similar to that of differentiation as opposed to exponentiation and is preferable from a computational perspective.

We will prove

Theorem 1.2. *There are infinitely many irreducible cubics $f \in \mathbb{Z}[x]$ with emergent reducibility at depth 1.*

E-mail address: jpreszler@member.ams.org.

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To do this, we show that the family

$$f_a(x) = -8ax^3 - (8a+2)x^2 + (4a-1)x + a$$

is irreducible for infinitely many integers a and that $f_a \circ f_a(x)$ is the product of a cubic and sextic polynomial for all a, namely

$$g_a(x) = 32a^2x^3 + (32a^2+16a)x^2 + (-16a^2+12a+2)x + (-4a^2-4a+1)$$

and

$$\begin{aligned} h_a(x) = 128a^2x^6 + (256a^2+32a)x^5 + 32ax^4 + (-160a^2-16a-4)x^3 \\ -(4a+2)x^2 + (16a^2+1)x + 2a^2. \end{aligned}$$

This family was merely the first identified by the author among many examples of non-monic integral cubics with emergent reducibility at depth 1. These examples were all found from a brute-force search using the FLiNT C library[3], with results double-checked in pari/gp[8].

2. The Irreducibility of f_a

Since reducible polynomials will have reducible iterates, it is crucial to show that infinitely many polynomials of the form $f_a(x)$ for $a \in \mathbb{Z}$ are irreducible. This can be accomplished, somewhat unsatisfactorily, by use of Hilbert's Irreducibility Theorem. We can consider our parameterized family as a family of polynomials in two variables, $f_a(x) = f(a,x) \in \mathbb{Q}[a,x]$. Since $f(a,x)$ is linear in a and the "coefficients" have no common factors in $\mathbb{Q}[x]$, we know that $f(a,x)$ is irreducible in $\mathbb{Q}[a,x]$. Hilbert's Irreducibility Theorem[7] ensures that for infinitely many values of $a \in \mathbb{Z}$ the specialization $f_a(x)$ is irreducible in $\mathbb{Q}[x]$. Unfortunately, this fails to give any specific values of a such that $f_a(x)$ is irreducible.

The following theorem gives a more effective determination of when $f_a(x)$ is irreducible. We note that computational evidence suggests that $f_a(x)$ is irreducible for every non-zero a (verified for $0 < |a| \leq 10^6$) but for our purposes it is sufficient to show that $f_a(x)$ is irreducible for infinitely many values of a.

Let $a \in \mathbb{Z}$ with $3 \nmid a$, then the polynomial

$$\begin{aligned} f_a(x) &= -8ax^3 - (8a+2)x^2 + (4a-1)x + a \\ &\equiv \begin{cases} x^3 + 2x^2 + 1 & \text{if } a \equiv 1 \mod (3) \\ -(x^3 + 2x + 1) & \text{if } a \equiv 2 \mod (3). \end{cases} \end{aligned} \tag{1}$$

Both polynomials are easily seen to be irreducible over $\mathbb{Z}/3$. Therefore, $f_a(x)$ is irreducible over $\mathbb{Z}$.

Considering $a = 3t$ for $t \in \mathbb{Z}$, reduction modulo other small primes shows that even more values of a will result in irreducible polynomials, matching computational evidence.

3. The Reducibility of $f_a \circ f_a$

The required reducibility will now be outlined and all calculations can be verified by hand, or easily with any computer algebra program. The composition of f_a with itself is

$$
\begin{aligned}
f_a \circ f_a(x) &= 4096a^4x^9 + (12288a^4 + 3072a^3)x^8 \\
&+(6144a^4 + 7680a^3 + 768a^2)x^7 + (-9728a^4 + 2560a^3 + 1408a^2 + 64a)x^6 \\
&+(-6144a^4 - 4864a^3 + 64a^2 + 32a)x^5 \\
&+(3072a^4 - 1920a^3 - 672a^2 - 80a - 8)x^4 \\
&+(1216a^4 + 1024a^3 - 64a^2 - 16a - 8)x^3 \\
&+(-192a^4 + 240a^3 + 40a^2 + 16a)x^2 \\
&+(-96a^4 - 40a^3 + 16a^2 - 4a + 1)x \\
&+(-8a^4 - 8a^3 + 2a^2).
\end{aligned} \tag{2}
$$

With the cubic $g_a(x)$ and the sextic $h_a(x)$ defined as

$$
\begin{aligned}
g_a(x) = 32a^2x^3 + (32a^2 + 16a)x^2 + (-16a^2 + 12a + 2)x \\
+(-4a^2 - 4a + 1)
\end{aligned} \tag{3}
$$

and

$$
\begin{aligned}
h_a(x) = 128a^2x^6 + (256a^2 + 32a)x^5 + 32ax^4 + (-160a^2 - 16a - 4)x^3 \\
-(4a + 2)x^2 + (16a^2 + 1)x + 2a^2
\end{aligned} \tag{4}
$$

then

$$g_a(x)h_a(x) = f_a \circ f_a(x). \tag{5}$$

Thus, $f_a \circ f_a$ is reducible and together with 2 we have shown:

Example 3.1. *For $3 \nmid a \in \mathbb{Z}$, the polynomials $f_a(x) = -8ax^3 - (8a+2)x^2 + (4a-1)x + a$ are irreducible with emergent reducibility of depth 1.*

Theorem 1.2 is a direct consequence of the existence of the family $f_a(x)$. We also note that computational evidence suggests g_a and h_a are irreducible over $\mathbb{Z}[x]$, but this in ancillary to our requirements. In the next section we show that there are a number of other examples of this behavior.

4. Other Examples

There are many examples of non-monic cubics with depth 1 emergent reducibility, likely including other parameterizable families. The more interesting situation is the case of monic integral cubics where there seems to be only a finite number with depth one emergent reducibility. The following is a list of all known examples where the absolute value of the coefficients are less than 500. These were found via an exhaustive brute-force search on an Intel i5-3337U 1.8GHz dual core (4 threads) running Linux using OpenMP and the FLiNT C library. The search took 1429 minutes for monic polynomials with the absolute value of all coefficients less than 500. With the absolute value of coefficients less than 100, the search is only 11 minutes.

$$x^3 \pm 9x^2 + 23x \pm 13 \tag{6}$$

$$x^3 \pm 6x^2 + 11x \pm 5 \tag{7}$$

$$x^3 \pm x^2 - 3x \mp 1 \tag{8}$$

$$x^3 \pm 4x^2 + 3x \mp 1 \tag{9}$$

This leads us to make the following conjecture:

Conjecture 4.1. *There are only finitely many monic cubics in $\mathbb{Z}[x]$ with depth one emergent reducibility.*

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