An Infinite Family of Cubic Polynomials with Depth 1 Emergent Reducibility

Jason I. Preszler

University of Puget Sound

jpreszler@pugetsound.edu

January 11, 2015

What is Emergent Reducibility?

We are interested in studying the **iterates**, $f(f(\dots f(x)\dots)) = f^{\circ n}(x)$, of polynomials $f(x)$.

1. If $f(x)$ is reducible, all iterates will remain reducible.
2. If $f(x)$ is irreducible, $f^{\circ n}(x)$ may become reducible at some $n \geq 1$.

Definition (Emergent Reducibility)

We say $f(x)$ has *emergent reducibility at depth* n if $f^{\circ k}(x)$ is irreducible for $0 \leq k \leq (n-1)$ and $f^{\circ n}(x)$ is reducible.

Note: Depth, n, tracks the number of composition operations done.

What is Emergent Reducibility?

Jason I. Preszler

We are interested in studying the **iterates**, $f(f(\dots f(x)\dots)) = f^{\circ n}(x)$, of polynomials $f(x)$.

1. If $f(x)$ is reducible, all iterates will remain reducible.
2. If $f(x)$ is irreducible, $f^{\circ n}(x)$ may become reducible at some $n \geq 1$.

Definition (Emergent Reducibility)

We say $f(x)$ has *emergent reducibility at depth* n if $f^{\circ k}(x)$ is irreducible for $0 \leq k \leq (n-1)$ and $f^{\circ n}(x)$ is reducible.

Note: Depth, n, tracks the number of composition operations done.

What is Emergent Reducibility?

We are interested in studying the **iterates**, $f(f(\ldots f(x)\ldots)) = f^{\circ n}(x)$, of polynomials $f(x)$.

1. If $f(x)$ is reducible, all iterates will remain reducible.
2. If $f(x)$ is irreducible, $f^{\circ n}(x)$ may become reducible at some $n \geq 1$.

Definition (Emergent Reducibility)

We say $f(x)$ has *emergent reducibility at depth* n if $f^{\circ k}(x)$ is irreducible for $0 \leq k \leq (n-1)$ and $f^{\circ n}(x)$ is reducible.

Note: Depth, n, tracks the number of composition operations done.

What is Emergent Reducibility?

We are interested in studying the **iterates**, $f(f(\dots f(x)\dots)) = f^{\circ n}(x)$, of polynomials $f(x)$.

1. If $f(x)$ is reducible, all iterates will remain reducible.
2. If $f(x)$ is irreducible, $f^{\circ n}(x)$ may become reducible at some $n \geq 1$.

Definition (Emergent Reducibility)

We say $f(x)$ has *emergent reducibility at depth* n if $f^{\circ k}(x)$ is irreducible for $0 \leq k \leq (n-1)$ and $f^{\circ n}(x)$ is reducible.

Note: Depth, n, tracks the number of composition operations done.

University of Puget Sound

Jason I. Preszler

What is Emergent Reducibility?

We are interested in studying the **iterates**, $f(f(\dots f(x)\dots)) = f^{\circ n}(x)$, of polynomials $f(x)$.

1. If $f(x)$ is reducible, all iterates will remain reducible.
2. If $f(x)$ is irreducible, $f^{\circ n}(x)$ may become reducible at some $n \geq 1$.

Definition (Emergent Reducibility)

We say $f(x)$ has *emergent reducibility at depth* n if $f^{\circ k}(x)$ is irreducible for $0 \leq k \leq (n-1)$ and $f^{\circ n}(x)$ is reducible.

Note: Depth, n, tracks the number of composition operations done.

Emergent Reducibility: Known Results

- The set of polynomials with emergent reducibility is **thin**.
- Odoni, [Odo85]: If f is p-Eisenstein then $f^{\circ n}$ is p-Eisenstein.
- R. Jones 2012 REU, [CCF+12]: There are finitely many quadratics with ER at depth $n \geq 2$ if certain conditions are met.
- [CCF+12] Iterates of quadratics that have ER will factor into 2 equal degree factors.
- Danielson and Fein, [DF02] proved that ER always occurs for $x^k - d$ if you pass to an appropriate extension and related ER of $x^k - d$ to Diophantine problems.
- [DF02] There are infinitely many $m \in \mathbb{Q}$ such that $x^2 + m$ has ER at depth 1.

Emergent Reducibility: Known Results

- The set of polynomials with emergent reducibility is **thin**.
- Odoni, [Odo85]: If f is p-Eisenstein then $f^{\circ n}$ is p-Eisenstein.
- R. Jones 2012 REU, [CCF$^+$12]: There are finitely many quadratics with ER at depth $n \geq 2$ if certain conditions are met.
- [CCF$^+$12] Iterates of quadratics that have ER will factor into 2 equal degree factors.
- Danielson and Fein, [DF02] proved that ER always occurs for $x^k - d$ if you pass to an appropriate extension and related ER of $x^k - d$ to Diophantine problems.
- [DF02] There are infinitely many $m \in \mathbb{Q}$ such that $x^2 + m$ has ER at depth 1.

Emergent Reducibility: Known Results

- The set of polynomials with emergent reducibility is **thin**.
- Odoni, [Odo85]: If f is p-Eisenstein then $f^{\circ n}$ is p-Eisenstein.
- R. Jones 2012 REU, [CCF$^+$12]: There are finitely many quadratics with ER at depth $n \geq 2$ if certain conditions are met.
- [CCF$^+$12] Iterates of quadratics that have ER will factor into 2 equal degree factors.
- Danielson and Fein, [DF02] proved that ER always occurs for $x^k - d$ if you pass to an appropriate extension and related ER of $x^k - d$ to Diophantine problems.
- [DF02] There are infinitely many $m \in \mathbb{Q}$ such that $x^2 + m$ has ER at depth 1.

Emergent Reducibility: Known Results

- The set of polynomials with emergent reducibility is **thin**.
- Odoni, [Odo85]: If f is p-Eisenstein then $f^{\circ n}$ is p-Eisenstein.
- R. Jones 2012 REU, [CCF$^+$12]: There are finitely many quadratics with ER at depth $n \geq 2$ if certain conditions are met.
- [CCF$^+$12] Iterates of quadratics that have ER will factor into 2 equal degree factors.
- Danielson and Fein, [DF02] proved that ER always occurs for $x^k - d$ if you pass to an appropriate extension and related ER of $x^k - d$ to Diophantine problems.
- [DF02] There are infinitely many $m \in \mathbb{Q}$ such that $x^2 + m$ has ER at depth 1.

Emergent Reducibility: Known Results

- The set of polynomials with emergent reducibility is **thin**.
- Odoni, [Odo85]: If f is p-Eisenstein then $f^{\circ n}$ is p-Eisenstein.
- R. Jones 2012 REU, [CCF$^+$12]: There are finitely many quadratics with ER at depth $n \geq 2$ if certain conditions are met.
- [CCF$^+$12] Iterates of quadratics that have ER will factor into 2 equal degree factors.
- Danielson and Fein, [DF02] proved that ER always occurs for $x^k - d$ if you pass to an appropriate extension and related ER of $x^k - d$ to Diophantine problems.
- [DF02] There are infinitely many $m \in \mathbb{Q}$ such that $x^2 + m$ has ER at depth 1.

Emergent Reducibility: Known Results

- The set of polynomials with emergent reducibility is **thin**.
- Odoni, [Odo85]: If f is p-Eisenstein then $f^{\circ n}$ is p-Eisenstein.
- R. Jones 2012 REU, [CCF$^+$12]: There are finitely many quadratics with ER at depth $n \geq 2$ if certain conditions are met.
- [CCF$^+$12] Iterates of quadratics that have ER will factor into 2 equal degree factors.
- Danielson and Fein, [DF02] proved that ER always occurs for $x^k - d$ if you pass to an appropriate extension and related ER of $x^k - d$ to Diophantine problems.
- [DF02] There are infinitely many $m \in \mathbb{Q}$ such that $x^2 + m$ has ER at depth 1.

Emergent Reducibility: Known Results

- The set of polynomials with emergent reducibility is **thin**.
- Odoni, [Odo85]: If f is p-Eisenstein then $f^{\circ n}$ is p-Eisenstein.
- R. Jones 2012 REU, [CCF+12]: There are finitely many quadratics with ER at depth $n \geq 2$ if certain conditions are met.
- [CCF+12] Iterates of quadratics that have ER will factor into 2 equal degree factors.
- Danielson and Fein, [DF02] proved that ER always occurs for $x^k - d$ if you pass to an appropriate extension and related ER of $x^k - d$ to Diophantine problems.
- [DF02] There are infinitely many $m \in \mathbb{Q}$ such that $x^2 + m$ has ER at depth 1.

$-8ax^3 - (8a+2)x^2 + (4a-1)x + a$: Part 1

Jason I. Preszler

Theorem ([Pre14])

For all $a \in \mathbb{Z}$, the cubic polynomial $f_a(x) = -8ax^3 - (8a+2)x^2 + (4a-1)x + a$ has iterate $f_a \circ f_a(x)$ that factors into the cubic and sextic with coefficients:

degree	Cubic	Sextic
0	$-4a^2 - 4a + 1$	$2a^2$
1	$-16a^2 + 12a + 2$	$16a^2 + 1$
2	$32a^2 + 16a$	$-4a - 2$
3	$32a^2$	$-160a^2 - 16a - 4$
4	0	$32a$
5	0	$256a^2 + 32a$
6	0	$128a^2$

$-8ax^3 - (8a+2)x^2 + (4a-1)x + a$: Part 1

Theorem ([Pre14])

For all $a \in \mathbb{Z}$, the cubic polynomial $f_a(x) = -8ax^3 - (8a+2)x^2 + (4a-1)x + a$ has iterate $f_a \circ f_a(x)$ that factors into the cubic and sextic with coefficients:

degree	Cubic	Sextic
0	$-4a^2 - 4a + 1$	$2a^2$
1	$-16a^2 + 12a + 2$	$16a^2 + 1$
2	$32a^2 + 16a$	$-4a - 2$
3	$32a^2$	$-160a^2 - 16a - 4$
4	0	$32a$
5	0	$256a^2 + 32a$
6	0	$128a^2$

$-8ax^3 - (8a+2)x^2 + (4a-1)x + a$: Part 2

Jason I. Preszler

Lemma ([Pre14])

The cubics $f_a(x)$ are irreducible over $\mathbb{Q}$ for $a \neq 0 \mod (3)$.

Computationally, $f_a(x)$ is irreducible for all $a \neq 0$ with $|a| \leq 10^6$.

Theorem ([Pre14])

There are infinitely many cubic polynomials in $\mathbb{Z}[x]$ with depth 1 emergent reducibility.

Other cubic families exist

$-8ax^3 - (8a+2)x^2 + (4a-1)x + a$: Part 2

Jason I. Preszler

Lemma ([Pre14])

The cubics $f_a(x)$ are irreducible over $\mathbb{Q}$ for $a \neq 0 \mod (3)$.

Computationally, $f_a(x)$ is irreducible for all $a \neq 0$ with $|a| \leq 10^6$.

Theorem ([Pre14])

There are infinitely many cubic polynomials in $\mathbb{Z}[x]$ with depth 1 emergent reducibility.

Other cubic families exist

$-8ax^3 - (8a+2)x^2 + (4a-1)x + a$: Part 2

Jason I. Preszler

Lemma ([Pre14])

The cubics $f_a(x)$ are irreducible over $\mathbb{Q}$ for $a \neq 0 \mod (3)$.

Computationally, $f_a(x)$ is irreducible for all $a \neq 0$ with $|a| \leq 10^6$.

Theorem ([Pre14])

There are infinitely many cubic polynomials in $\mathbb{Z}[x]$ with depth 1 emergent reducibility.

Other cubic families exist

$-8ax^3 - (8a+2)x^2 + (4a-1)x + a$: Part 2

Jason I. Preszler

Lemma ([Pre14])

The cubics $f_a(x)$ are irreducible over $\mathbb{Q}$ for $a \neq 0 \mod (3)$.

Computationally, $f_a(x)$ is irreducible for all $a \neq 0$ with $|a| \leq 10^6$.

Theorem ([Pre14])

There are infinitely many cubic polynomials in $\mathbb{Z}[x]$ with depth 1 emergent reducibility.

Other cubic families exist

$-8ax^3-(8a+2)x^2+(4a-1)x+a$: Part 2

Jason I. Preszler

Lemma ([Pre14])

The cubics $f_a(x)$ are irreducible over $\mathbb{Q}$ for $a \not\equiv 0 \mod (3)$.

Computationally, $f_a(x)$ is irreducible for all $a \neq 0$ with $|a| \leq 10^6$.

Theorem ([Pre14])

There are infinitely many cubic polynomials in $\mathbb{Z}[x]$ with depth 1 emergent reducibility.

Other cubic families exist

Monic Cubic Examples

All have emergent reducibility at depth 1

- $x^3 \pm 9x^2 + 23x \pm 13$
- $x^3 \pm 6x^2 + 11x \pm 5$
- $x^3 \pm x^2 - 3x \mp 1$
- $x^3 \pm 4x^2 + 3x \mp 1$

In all cases $f \circ f$ factors as cubic and 6^{th} deg. poly.

Monic Cubic Examples

All have emergent reducibility at depth 1

- $x^3 \pm 9x^2 + 23x \pm 13$
- $x^3 \pm 6x^2 + 11x \pm 5$
- $x^3 \pm x^2 - 3x \mp 1$
- $x^3 \pm 4x^2 + 3x \mp 1$

In all cases $f \circ f$ factors as cubic and 6^{th} deg. poly.

Monic Quartic Examples

All have emergent reducibility at depth 1

- $x^4 - 8x^3 + 13x^2 + 12x + 1$
- $x^4 - 5x^3 + 5x^2 + 3x - 1$
- $x^4 - 2x^3 - 2x^2 + 3x + 1$
- $x^4 - 7x^2 + 13$
- $x^4 + 3x^3 - x + 1$
- and more

Factors have degrees $(8, 8)$ or $(4, 12)$

Monic Quartic Examples

Jason I. Preszler

All have emergent reducibility at depth 1

- $x^4 - 8x^3 + 13x^2 + 12x + 1$
- $x^4 - 5x^3 + 5x^2 + 3x - 1$
- $x^4 - 2x^3 - 2x^2 + 3x + 1$
- $x^4 - 7x^2 + 13$
- $x^4 + 3x^3 - x + 1$
- and more

Factors have degrees $(8, 8)$ or $(4, 12)$

$-ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$: Part 1

Theorem ([Pre15])

For all $a \in \mathbb{Z}$, the quartic polynomial $g_a(x) = -ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$ satisfies $g_a \circ g_a(x) = h(x)k(x)$ with coefficients:

degree	$h(x)$	$k(x)$
0	$-a(a-2)$	$a^3 - a - 1$
1	$(a-1)(4a+1)$	$-4a^3 - 3a^2 - a - 1$
2	$-(a+1)(2(a+1)-1)$	$2a^3 + 3a^2 + 2a - 1$
3	$-a(8a-1)$	$a(8a^2 + 9a + 6)$
4	$a(5a+3)$	$-a(5a^2 + 3a + 3)$
5	$a(8a+3)$	$-a^2(8a+9)$
6	$-a(2a-1)$	$a^2(2a-3)$
7	$-4a^2$	$4a^3$
8	$-a^2$	a^3

$-ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$: Part 1

Theorem ([Pre15])

For all $a \in \mathbb{Z}$, *the quartic polynomial* $g_a(x) = -ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$ *satisfies* $g_a \circ g_a(x) = h(x)k(x)$ *with coefficients:*

degree	$h(x)$	$k(x)$
0	$-a(a-2)$	$a^3 - a - 1$
1	$(a-1)(4a+1)$	$-4a^3 - 3a^2 - a - 1$
2	$-(a+1)(2(a+1)-1)$	$2a^3 + 3a^2 + 2a - 1$
3	$-a(8a-1)$	$a(8a^2 + 9a + 6)$
4	$a(5a+3)$	$-a(5a^2 + 3a + 3)$
5	$a(8a+3)$	$-a^2(8a+9)$
6	$-a(2a-1)$	$a^2(2a-3)$
7	$-4a^2$	$4a^3$
8	$-a^2$	a^3

$-ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$: Part 2

Jason I. Preszler

Theorem ([Pre15])

The quartics $g_a(x) = -ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$ are irreducibile over $\mathbb{Q}$ for $a \geq 1$ if and only if a is not an oblong number ($a_n = n(n+1)$, OEIS A002378).

Corollary ([Pre15])

There are infinitely many integer quartics with depth 1 emergent reducibility.

$-ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$: Part 2

Theorem ([Pre15])

The quartics $g_a(x) = -ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$ are irreducibile over $\mathbb{Q}$ for $a \geq 1$ if and only if a is not an oblong number ($a_n = n(n+1)$, OEIS A002378).

Corollary ([Pre15])

There are infinitely many integer quartics with depth 1 emergent reducibility.

$-ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$: Part 2

Theorem ([Pre15])

The quartics $g_a(x) = -ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$ are irreducibile over $\mathbb{Q}$ for $a \geq 1$ if and only if a is not an oblong number ($a_n = n(n+1)$, OEIS A002378).

Corollary ([Pre15])

There are infinitely many integer quartics with depth 1 emergent reducibility.

References

K. Chamberlin, E. Colbert, S. Frechette, P. Hefferman, R. Jones, and S. Orchard, *Newly reducible iterates in families of quadratic polynomials*, ArXiv e-prints (2012).

Lynda Danielson and Burton Fein, *On the irreducibility of the iterates of $x^n - b$*, Proc. Amer. Math. Soc. **130** (2002), no. 6, 1589–1596 (electronic).

R.W.K. Odoni, *The galois theory of iterates and composites of polynomials*, Proc. London Math. Soc. **51** (1985), no. 3, 385–414 (electronic).

J. I. Preszler, *An Infinite Family of Cubics with Emergent Reducibility at Depth 1*, ArXiv e-prints (2014).

________, *An Infinite Family of Quartics with Depth 1 Emergent Reducibility Related to the Oblong Numbers*, In Preparation (2015).