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An Infinite Family of Cubic Polynomials with Depth 1 Emergent Reducibility

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We are interested in studying the **iterates**, $f(f(\ldots f(x) \ldots)) = f^{\circ n}(x)$, of polynomials f(x).

If f(x) is reducible, all iterates will remain reducible.
If f(x) is irreducible, f^{on}(x) may become reducible at some n > 1

Definition (Emergent Reducibility)

We say f(x) has emergent reducibility at depth n if $f^{\circ k}(x)$ is irreducible for $0 \le k \le (n-1)$ and $f^{\circ n}(x)$ is reducible.

Note: Depth, n, tracks the number of composition operations done.



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- The set of polynomials with emergent reducibility is **thin**.
- Odoni, [Odo85]: If f is p-Eisenstein then f^{on} is p-Eisenstein.
- R. Jones 2012 REU, [CCF⁺12]: There are finitely many quadratics with ER at depth $n \ge 2$ if certain conditions are met.
- [CCF⁺12] Iterates of quadratics that have ER will factor into 2 equal degree factors.
- Danielson and Fein, [DF02] proved that ER always occurs for $x^k - d$ if you pass to an appropriate extension and related ER of $x^k - d$ to Diophantine problems.
- [DF02] There are infinitely many $m \in \mathbb{Q}$ such that $x^2 + m$ has ER at depth 1.



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 $-8ax^3 - (8a+2)x^2 + (4a-1)x + a$: Part 1

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For all $a \in \mathbb{Z}$, the cubic polynomial $f_a(x) = -8ax^3 - (8a+2)x^2 + (4a-1)x + a$ has iterate $f_a \circ f_a(x)$ that factors into the cubic and sextic with coefficients:

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$$-8ax^3 - (8a+2)x^2 + (4a-1)x + a$$
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Theorem ([Pre14])

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degree	Cubic	Sextic
0	$-4a^2 - 4a + 1$	$2a^2$
1	$-16a^2 + 12a + 2$	$16a^2 + 1$
2	$32a^2 + 16a$	-4a - 2
3	$32a^2$	$-160a^2 - 16a - 4$
4	0	32a
5	0	$256a^2 + 32a$
6	0	$128a^{2}$



 $-8ax^3 - (8a+2)x^2 + (4a-1)x + a$: Part 2

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Lemma ([Pre14])

he cubics $f_a(x)$ are irreducible over \mathbb{Q} for $a \neq 0 \mod (3)$.

Computationally, $f_a(x)$ is irreducible for all $a \neq 0$ with $|a| \leq 10^6.$

Theorem ([Pre14])

There are infinitely many cubic polynomials in $\mathbb{Z}[x]$ with depth 1 emergent reducibility.

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All have emergent reducibility at depth 1

• $x^3 \pm 9x^2 + 23x \pm 13$

- $x^3 \pm 6x^2 + 11x \pm 5$
- $x^3 \pm x^2 3x \mp 1$
- $x^3 \pm 4x^2 + 3x \mp 1$

In all cases $f \circ f$ factors as cubic and 6^{th} deg. poly.

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All have emergent reducibility at depth 1

• $x^4 - 8x^3 + 13x^2 + 12x + 1$

•
$$x^4 - 5x^3 + 5x^2 + 3x - 1$$

•
$$x^4 - 2x^3 - 2x^2 + 3x + 1$$

•
$$x^4 - 7x^2 + 13$$

•
$$x^4 + 3x^3 - x + 1$$

and more

Factors have degrees (8,8) or (4,12)

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$$-ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$$
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For all $a \in \mathbb{Z}$, the quartic polynomial $g_a(x) = -ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$ satisfies $g_a \circ g_a(x) = h(x)k(x)$ with coefficients:

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degree	h(x)	k(x)
0	-a(a-2)	$a^3 - a - 1$
1	(a-1)(4a+1)	$-4a^3 - 3a^2 - a - 1$
2	-(a+1)(2(a+1)-1)	$2a^3 + 3a^2 + 2a - 1$
3	-a(8a-1)	$a(8a^2 + 9a + 6)$
4	a(5a + 3)	$-a(5a^2+3a+3)$
5	a(8a + 3)	$-a^2(8a+9)$
6	-a(2a-1)	$a^2(2a-3)$
7	$-4a^{2}$	$4a^{3}$
8	$-a^2$	a^3



 $-ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$: Part 2

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Theorem ([Pre15])

The quartics $g_a(x) = -ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$ are irreducibile over \mathbb{Q} for $a \ge 1$ if and only if a is not an oblong number $(a_n = n(n+1), \text{ OEIS A002378}).$

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Corollary ([Pre15])

There are infinitely many integer quartics with depth 1 emergent reducibility.



 $-ax^4 - 2ax^3 + (a+1)x^2 + (2a+1)x - a$: Part 2

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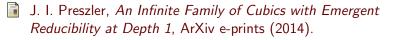
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